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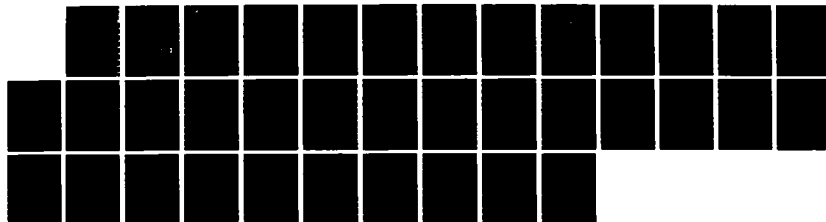
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**ASYMPTOTICALLY EXACT A-POSTERIORI ERROR ESTIMATOR
FOR BIQUADRATIC ELEMENTS**

by

I. Babuška

Institute for Physical Science and Technology
University of Maryland, College Park, Maryland 20742
U.S.A.

and

Dehao Yu

Computing Center Academia Sinica
Beijing

The People's Republic of China

and

Institute for Physical Science and Technology
University of Maryland, College Park, Maryland 20742
U.S.A.

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I. Babuška¹

Institute for Physical Science and Technology
University of Maryland, College Park, MD 20742
U.S.A.

Dehao Yu²

Computing Center Academia Sinica
Beijing

The People's Republic of China
and

Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742
U.S.A.

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SUMMARY

The paper addresses the finite element method with a-posteriori error estimation for elements of degree $p = 1$ and $p = 2$. It gives the formulae for the error indicators and error estimators. Basic mathematical characterization of the estimators are given and it is shown that the estimators for $p = 1$ and $p = 2$ have different structures. Numerical examples show the effectivity of the approach and the high quality of the estimator.

1. INTRODUCTION

During the last few years a significant progress has been achieved in the theory and the implementation of the adaptive procedures and a-posteriori error estimation for the finite element method. For the survey of the today's state of the art of the quality assessment of finite element solutions we refer to [1]. Adaptive finite element codes FEARS [2], PLTGM [3], EXPDES [4] were developed. For the theoretical and implementational aspects related to the mentioned codes we refer to [5] to [10]. Various approaches for obtaining a-posteriori error estimates were recently developed. See, for example, [1] - [15] and other references mentioned in [1].

The codes and the theory mentioned above are for the finite element method with elements of degree one. In this paper we are addressing the questions arising from the use of elements of degree two. We will concentrate on the comparison between the approaches based on the elements of degree one and two.



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2. THE MODEL PROBLEM

Let us consider the model problem of the Laplace equation

$$(2.1.) \quad L(u) = -\Delta u = f \text{ on } \Omega \subset \mathbb{R}^2$$

with the boundary conditions

$$(2.2a) \quad u = 0 \text{ on } \partial^0 \Omega$$

$$(2.2b) \quad \frac{\partial u}{\partial n} = t \text{ on } \partial^1 \Omega$$

where n is the outward normal to $\partial\Omega$.

The solution u_0 of the problem (2.1) (2.2) is understood in the standard weak sense: $u_0 \in H_0^1(\Omega) = \{u | u \in H^1(\Omega), u = 0 \text{ on } \partial^0 \Omega\}$ such that

$$(2.3) \quad B(u_0, v) = \int_{\Omega} (\nabla u_0 \cdot \nabla v) dx = \int_{\Omega} f v dx + \int_{\partial^1 \Omega} t v ds$$

holds for all $v \in H_0^1(\Omega)$. By $H^1(\Omega)$ we denote the standard Sobolev space. By $\|u\|_E^2 = B(u, u)$ we denote the energy norm of function $u \in H_0^1(\Omega)$.

Remark. Although we restricted ourself to a very special case (2.1), (2.2) the theory and conclusions hold for the general elliptic equation on bounded domains in \mathbb{R}^2 .

We will consider two illustrative model problems.

Problem 1. The domain Ω and the boundary conditions are shown in Fig. 2.1.

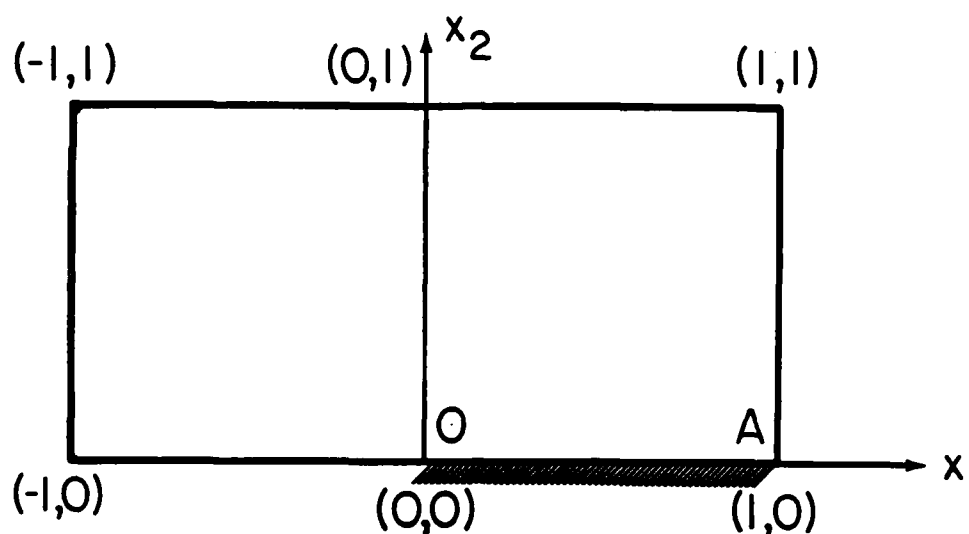


Fig. 2.1. Scheme of Problem 1.

$\partial^0\Omega$ consists of OA and on $\partial^1\Omega$ t is defined so that the exact solution u_0 is

$$u_0 = 0.0700754r^{1/2} \sin \theta/2$$

and

$$|u_0|_E = 0.657878 \times 10^{-1}.$$

By (r, θ) we denote the polar coordinates.

Problem 2. The (slit) domain Ω and the boundary conditions are shown in Fig. 2.2.

$\partial^0\Omega$ consists of upper side OA (marked) and on $\partial^1\Omega$, t is defined so that the exact solution u_0 is

$$u_0 = 0.0635659r^{1/4} \sin \theta/4$$

$$|u_0|_E = 0.579428 \times 10^{-1}$$

Problem 1 and 2 have obviously solution with different strength of singularity.

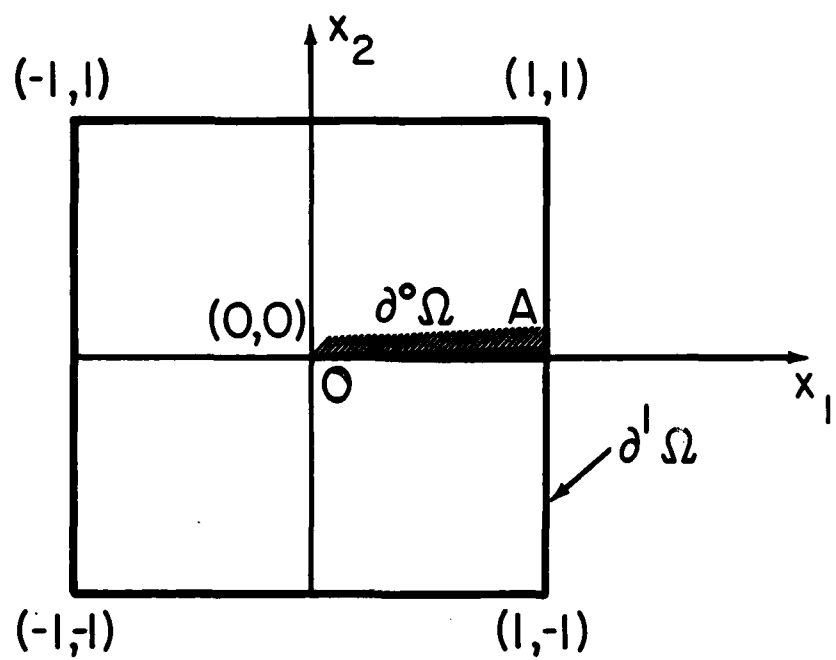


Fig. 2.2. Scheme of Problem 2.

3. THE FINITE ELEMENT METHOD

We will consider the finite element method with the bilinear and biquadratic elements on the meshes which are used in the Program FEARS. These meshes have regular and irregular nodal points. The values of the finite element solution in the irregular nodal points is determined by the requirement that the elements are conforming. Fig. 3.1a,b show the meshes with 20 elements for the model problem 1 and bilinear and biquadratic elements. The regular nodal points are marked by \cdot , while the irregular points are marked by \times . We will not introduce here a formal definition and refer to [2], [6], [7].

We have shown the meshes only for the special case. They are in general defined for domains with curved boundaries. See [2], [6].

The finite element solution u_{FE} is defined in the standard way. By e we denote the error of the finite element solution u_{FE} , $e = u_{FE} - u$.

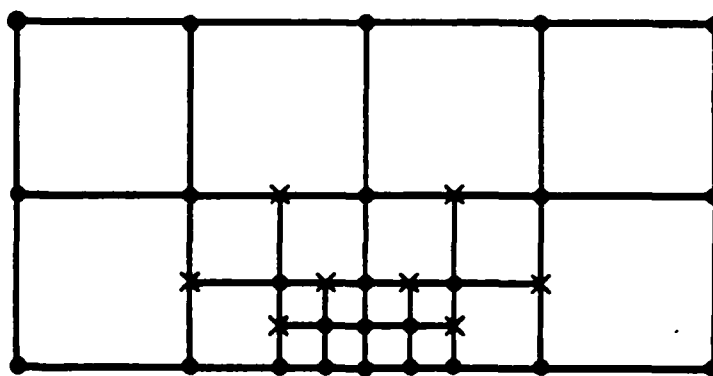


Fig. 3.1a. The regular and irregular nodal points for $p = 1$.

4. THE ADAPTIVE FINITE ELEMENT METHOD A-POSTERIORI ERROR ESTIMATION AND THE QUALITY ASSESSMENT

We will distinguish between the feedback and adaptive finite element method. A feedback method utilizes the currently available information for steering the computational process. The adaptive approach is a feedback approach which has clearly defined optimal properties. Hence the assessment of the quality of a feedback approach is relative to the criterion of the optimality. We will not go here into details but refer to [16], [17].

We will say that the feedback is adaptive with respect to the convergence rate measure if the rate is the same as the theoretically best one among all the meshes with the same number of degrees of freedom N and the same degrees of elements p .

For our model problems the feedback approach will be adaptive if the (asymptotic) rate is $O(N^{-1/2})$ for the elements of degree $p = 1$ and is $O(N^{-1})$ for the elements of degree $p = 2$ in both cases, i.e. the rate is independent of the strength of the singularity. We note that the same rate is obtained when the solution is smooth.

The quality of the a-posteriori error estimator ϵ is measured by its effectivity index $\theta = \epsilon / |e|_E$ where $|e|_E$ is the energy norm of the error of the finite element solution. In practice we have to require that $|\theta - 1| \leq .2$ (say) when the accuracy of u_{FE} is in the range of the engineering requirement ($\leq 10\%$ say). In addition we prefer that $\theta \geq 1$ so that the true error is overestimated rather than underestimated. This is important because the error estimator is used as the stopping criterion. Obviously if $\theta > 2$ or $\theta < 1/2$, then the error estimator is practically unacceptable.

5. THE ERROR INDICATORS AND ESTIMATORS

The error estimator ϵ is computed through the error indicators $\eta(\Delta)$ of the single element Δ . Then the error estimator (which approximate $|e|_E$) is given by

$$(5.1) \quad \epsilon^2 = \sum_{\Delta} \eta(\Delta)^2$$

where the sum is taken over all elements.

The error indicators are also used for the steering the feedback finite element method. We refer to [18] for the detailed analyses of various steering strategies. The basic strategy is to refine all elements having the error indicators over the threshold τ , for example, $\tau = \gamma \max \eta(\Delta)$ where $0 < \gamma \leq 1$ is a-priori chosen and the maximum is taken over all elements.

The code FEARS uses basically this type of strategy but more sophisticated to minimize computer time by avoiding repeated recomputation. The experience with FEARS shows that the total computer time is 2-3 times the computer time solving the final mesh only.

We note that error indicators which lead to completely unacceptable quality of the error estimators can still perform relatively well in steering the feedback approach.

Let us now define the error indicators for the bilinear and biquadratic elements. For simplicity of the exposition, we will consider only an uniform patch of the mesh. The data management of the adaptive solver is based on the tree structure (see [7]) and mesh is defined in a recursive way. Hence, every element has a "father" which has four

"sons." Fig. 5.1 shows uniform mesh with element Δ_i^0 , $i = 1, 2, 3, 4$ belonging to one "father" (Δ^0) shown also in the figure.

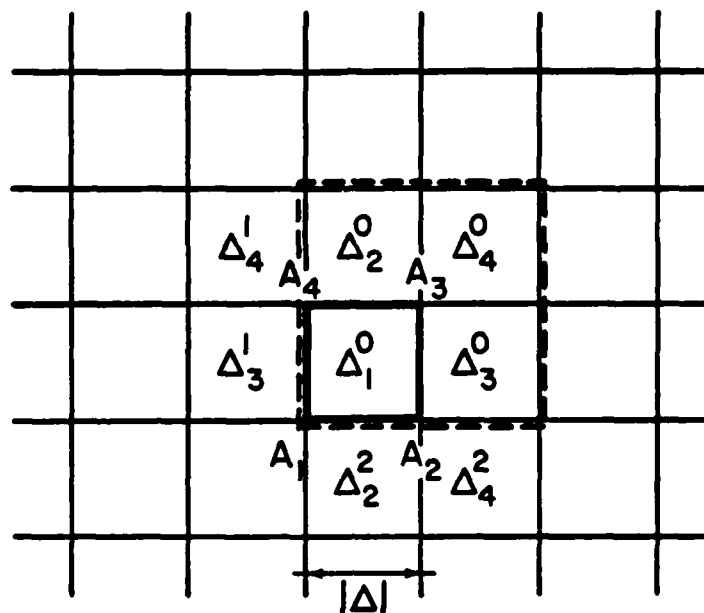


Fig. 5.1. The element and his "father".

A. Error indicator for the bilinear elements.

We will define the error indicator as used in FEARS but simplified for the special problem under consideration.

Let

$$(5.2a) \quad \eta_1^2(\Delta_1^0) = |\Delta|^2 \sum_{j=1}^2 \left(\max_{i=1, \dots, 4} r_{x_j}(A_i) \right)^2$$

$$(5.2b) \quad \eta_2^2(\Delta_1^0) = |\Delta|^2 \|\tilde{R}\|_{L_2(\Delta_1^0)}^2$$

where $|\Delta|$ is the length of the element side, $r_{x_j}(A_j)$ is the jump of the derivative in the x_j direction in the vertex A_j of the element Δ_1^0 and

$$(5.3a) \quad \bar{R} = f + \Delta u_{FE} - p$$

where p is such that $\int_{\Delta_1^0} \bar{R} \, dx = 0$.

Now we define

$$(5.4a) \quad \eta^2(\Delta_1^0) = \eta_1^2(\Delta_1^0) + \eta_2^2(\Delta_1^0).$$

It can be proven (see [6]) that $\eta_2(\Delta_1^0) \ll \eta_1(\Delta_1^0)$ and hence it can be neglected and define

$$(5.4b) \quad \bar{\eta}^2(\Delta_1^0) = \eta_1^2(\Delta_1^0).$$

Experience with FEARS shows that in fact the use of η and $\bar{\eta}$ practically does not change the estimator in [6].

Remark 1. If the mesh is uniform and the solution is quadratic, then the error estimator is exact.

Remark 2. We could use other equivalent estimators, for example, replace η_1^2 by the integral of the square of the jump of the derivative of the finite element solution over the boundary of Δ_1^0 (multiplied by $|\Delta|$). Obviously $\eta_1^2(\Delta_1^0)$ is the value of this integral computed by a quadrature. Nevertheless, experience shows that for small accuracy (large error) the estimator based on the integral tends to underestimate the error and hence the definition of the indicator η_1 corrects this undesirable and enlarges the range of the asymptotic validity of the situation of the estimator (see also numerical results in section 7).

B. Error indicator for biquadratic elements

Let us assume that the center of the element lies in the origin.

Define

$$\delta_i = x_i^2 - \frac{|\Delta|^2}{4}, \quad i = 1, 2$$

$$v_i = x_i \delta_1 \delta_2$$

$$(5.5a) \quad G(v_i) = \int_{\Delta_1^0} (f v_i - \nabla u_{FE} \cdot \nabla v_i) dx, \quad i = 1, 2$$

$$(5.5b) \quad b_i = -\frac{120}{|\Delta|^8} G(v_i), \quad i = 1, 2$$

$$(5.6) \quad \eta_1^2(\Delta_1^0) = \frac{720}{|\Delta|^{10}} \sum_{i=1}^2 (G(v_i))^2.$$

(The integrals in (5.5a) are numerically computed.)

$$(5.7) \quad \eta_2^2(\Delta_1^0) = |\Delta| \sum_{j=1}^2 \int (r_{x_j})^2 ds$$

where the integral is taken over the sides of the elements perpendicular to x_j and r_{x_j} are as before the jump in the derivatives.

$$(5.8) \quad \eta_3^2(\Delta_1^0) = |\Delta|^2 \|\bar{R}\|_{L_2(\Delta_1^0)}^2$$

where

$$\bar{R} = f + \Delta U$$

$$U = u_{FE} + b_1 x_1 \delta_1 + b_2 x_2 \delta_2$$

and b_i are given by (5.5b). Then we define

$$\eta^2(\Delta_1^0) = \sum_{i=1}^3 \eta_i^2(\Delta_1^0)$$

and ϵ is defined by (5.1).

It can be shown, see [19], that if the solution is sufficiently smooth, then η_2 and η_3 can be neglected with respect to η_1 and hence we can use the indicator

$$\bar{\eta}(\Delta_1^0) = \eta_1(\Delta_1^0)$$

as in the case for $p = 1$. Nevertheless, we see a significant difference, namely that in the case $p = 1$ the principal part of the indicator is the "boundary line part" (jumps), while for $p = 2$ it is the "volume" part. From the implementational point of view, the "volume" indicator is more advantageous.

In the case of the smooth solution u_0 and reasonable meshes the indicators for $p = 1$ and $p = 2$ are of the same magnitude and the estimators are of the high quality, namely

$$|e|_E = \epsilon(1 + \theta(\epsilon)),$$

where the term $\theta(\epsilon)$ is very small. Nevertheless, if the solution is unsmooth as in our model problems, then the indicators are of very different magnitude when the elements are adjacent to the origin where the solution is singular and the error is in these elements strongly underestimated. This is especially important in the case $p = 2$ because the error indicators in the area where the solution is smooth are much smaller than for $p = 1$. This makes the contribution of the largest indicators to the error estimator much larger.

The elements which are in the area of the singularity of the solution are refined during the adaptive mesh construction and are located in the terminal points of the tree data structure. These elements are quadruples of the elements belonging to one "father". Fig. 5.2 shows in the left corner such a quadruple and its "father" and "grandfather".

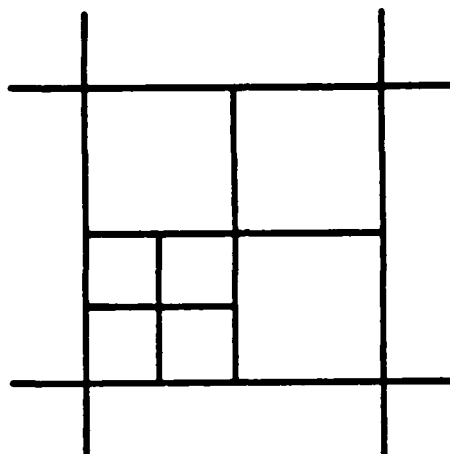


Fig. 5.2. Scheme of the terminal element.

Hence, we will modify the error indicators in these terminal quadruples in dependence on the distribution of the error indicators. (This modification can be implemented in a very effective way.)

The modified estimators $\hat{\eta}$ are constructed from η as follows: the four indicators of the quadruple of the elements are ordered by the magnitude. For the sake of the explanation, assume that $\eta^2(\Delta_1^0) \geq \eta^2(\Delta_3^0) \geq \eta^2(\Delta_2^0) \geq \eta^2(\Delta_4^0)$ (see Fig. 5.1).

If $\eta^2(\Delta_1^0) \geq \gamma \eta^2(\Delta_3^1)$ with $\gamma = 8$, then

$$\hat{\eta}^2(\Delta_1^0) = \eta^2(\Delta_1^0) F(R)$$

$$\hat{\eta}^2(\Delta_i^0) = \eta^2(\Delta_i^0),$$

$$i = 2, 3, 4$$

and

$$\begin{aligned}
 F(R) &= 0.031605R + 1 && \text{for } 0 \leq R \leq 234 \\
 &= 3.4813e^{3.762 \times 10^{-3}R} && \text{for } 234 \leq R \leq 1000 \\
 &= 150 && \text{for } R > 1000
 \end{aligned}$$

with $R = \frac{\eta^2(\Delta_1^0)}{\eta^2(\Delta_4^0)}$ where Δ_4^0 is the diagonal element to Δ_1^0 . If $\eta^2(\Delta_1^0) < \gamma\eta^2(\Delta_3^0)$ and $\eta^2(\Delta_3^0) \geq \beta\eta^2(\Delta_2^0)$ with $\beta = 100$, then

$$\hat{\eta}^2(\Delta_1^0) = G(R_1)\eta^2(\Delta_1^0)$$

$$\hat{\eta}^2(\Delta_3^0) = G(R_3)\eta^2(\Delta_3^0)$$

$$\hat{\eta}^2(\Delta_2^0) = \eta^2(\Delta_2^0)$$

$$\hat{\eta}^2(\Delta_4^0) = \eta^2(\Delta_4^0)$$

where

$$\begin{aligned}
 G(R) &= 0.003499R + 1 && \text{for } R \leq 700 \\
 &= 1.7132e^{R \cdot 10^{-3}} && \text{for } 700 \leq R \leq 4000 \\
 &= 93.54 && \text{for } 4000 \leq R
 \end{aligned}$$

and

$$R_1 = \frac{\eta^2(\Delta_1^0)}{\eta^2(\Delta_4^0)}, \quad R_3 = \frac{\eta^2(\Delta_3^0)}{\eta^2(\Delta_2^0)}$$

(where as before in the denominator is the indicator of the diagonally opposite element). If $\eta^2(\Delta_3^0) < \beta\eta^2(\Delta_2^0)$, then

$$\hat{n}^2(\Delta_i^0) = n^2(\Delta_i^0), \quad i = 1, 2, 3, 4.$$

The above formulae are characterizing the singularity of a point character (F(R)) (e.g. of the type occurring in our model problems) or the line singularity along the element side (G(R)). The formulae were developed by analyzing theoretically and numerically these cases and their validity was tested in a set of computations.

6. SOME BASIC PROPERTIES OF THE ESTIMATORS

Let us mention the essential properties of the estimator. We list the basic assumptions but not completely accurately because their complexity. We refer to [6] and [19] to precise formulations.

a) The mesh is a K-mesh. Roughly speaking, the mesh is a K-mesh if there are not too many irregular nodal points between the regular ones.

b) The mesh is nearly equilibrated. This means that the error indicators are nearly equal (except possibly small number of exceptions).

c) $|e|_E \geq Ch^p$ where h is the maximal element size. The conditions (a,b) usually lead to c.

d) The meshes are patchwise uniform. This roughly means that large pieces of meshes have uniform character.

e) The mesh is such that (roughly speaking) the major part of the error is in the areas where the solution has no singularity (due to proper refinement) (usually b leads to this assumption).

Under the above assumptions, the following theorem holds (see [19]).

Theorem 6.1. For any $\tau > 0$ and $h \leq h_0(\tau)$, τ sufficiently small

$$|\theta - 1| \leq M\tau$$

where the constant M is independent of τ .

Theorem 6.1 is valid for $p = 1$ and $p = 2$. Also for not modified estimator. Nevertheless, $M\tau$ could be small only for very small h , i.e. for very high accuracy which is not practically relevant. Hence the modified estimator is, as we will see, essential for practical computation.

7. NUMERICAL EXAMPLES

We will illustrate the performance of the feedback finite element method and the error estimator based on the error indicators introduced in previous sections. Let us summarize typical behavior one can observe

1) Fig. 7.1 shows the values of the modified error indicators $\eta^2(\Delta)$ for Problem 1, biquadratic elements ($p=2$) and various meshes with M elements and N degrees of freedom.

All indicators are of the same magnitude (10^{-8}) except the ones adjoined to the singularity point which are of much larger magnitude (10^{-5}).

The indicators of the elements which are not refined do not change significantly by the refinement of other elements.

2) The maximal error indicator is decreased by the factor $\frac{1}{2}$ for problem 1 (by $\sqrt{2}/2$ for problem 2) when the element is refined by dividing it into four elements. This factor is directly related to the strength of the singularity ($r^{1/2}$, $r^{1/4}$) of the solution. (Note that this factor is independent of p). For $p = 2$ and problem 1 the total error ϵ is essentially the maximal error indicator until the indicator is of the order 10^{-7} which occurs when the smallest element is of order 2^{-10} , i.e. of the order $\frac{1}{1000}$ and the error $|e|_E$ is of the magnitude 1% of $|u|_E$. Hence the rate is exponential. Fig. 7.2a, respectively 7.2b, shows in the semilogarithmic resp. double logarithmical scale the behavior of the error $|e|_E$. Similar behavior can be observed in the case of problem 2. See Fig. 7.3a,b. In Fig. 7.2b and 7.3b the slope of the maximal asymptotic rate $N^{-1/2}$ and N^{-1} for the elements of degree $p = 1$ and $p = 2$ is shown. This rate is achieved for adaptively constructed meshes for high accuracies independently of the strength of

the singularity. We see that for $p = 1$ this rate is achieved in the range of engineering accuracy while for $p = 2$ the rate is in this range still exponential and the asymptotic rate N^{-1} is achieved for much higher accuracies.

3) When the indicator of the element adjoined to the origin will be of order of the other indicators, then many elements will be refined. The indicators $\eta^2(\Delta)$ in the area where the solution is smooth will decrease by the magnitude $2^6 = 64$ for $p = 2$ and $2^4 = 16$ for $p = 1$. This refinement is then followed by a series of refinement of the elements adjoined to the singularity. Fig. 7.2 shows clearly this character for $p = 1$. For $p = 2$ the refinement of many elements did not occur in the figure.

4) The refinement by the bisection procedure is not an optimal one for the convergence. Nevertheless, it is very advantageous for the data treatment. We refer to [20] for the analyses of the optimal refinement in one dimension.

5) An effective feedback procedure has (especially for $p = 2$) anticipated the character of the proper refinement so that the proper mesh is designed quickly. It is obvious that the refinement character can be recognized. In the program FEARS this procedure is called a short pass procedure.

6) The quality of the error estimator is high. Fig. 7.2 shows the error and the error indicator for $p = 2$. In Table 7.1 we show typical data for the meshes with 8-26 elements for Problem 1. We clearly see that the effectivity index for the nonmodified estimator is unacceptable although it increases with $\|e\|_E \rightarrow 0$ (and theoretically converges to 1). On the other hand, the modified estimator has effectivity index of high quality.

7) Comparing performance of the bilinear ($p = 1$) and biquadratic mesh, we see that in the presence of a singularity the bilinear elements perform better for low accuracy while higher accuracy cannot be practically achieved by bilinear elements. If the solution is smooth, then the biquadratic elements perform better in larger range of the accuracy (see also Fig. 7.3a,b). The performance of biquadratic elements could be improved if stronger refinement than based on bisection could be effectively made.

M=8 N = 40

2.18-8	8.12-8	2.23-8	1.26-8
5.71-8	3.01-4	2.66-5	4.62-8

M=14 N = 60

1.09-8	4.92-8		4.80-8		1.03-7
5.27-8	2.09-8	4.13-8	1.11-8	7.11-9	4.45-8
	3.15-8	5.91-5	1.23-5	2.09-8	

Fig. 7.1a. Values of the modified error indicators $\eta^2(\Delta)$ for $M = 8, 14$.

M = 20		N = 80									
1.05-8		4.87-8		4.99-8		1.03-8					
5.12-8		1.24-8		2.82-8		2.67-8		5.89-9		4.73-8	
		3.54-8						1.94-8			

1.03-8		2.09-8		5.68-9		3.51-9	
1.60-8		3.05-5		6.22-6		1.05-8	

DETAIL

Fig. 7.1b. Values of the modified error indicators $\eta^2(\Delta)$ for $M = 20$.

M = 26 N = 100

1.04-8	4.86-8		5.04-8		1.03-8
5.02-8	1.20-8	2.74-8	2.75-8	5.99-9	4.88-8
	3.45-8			2.05-8	

6.09-9	1.42-8		1.34-8		2.92-9
1.79-8	8.07-9	1.05-8	2.85-9	1.77-9	9.74-9
	5.18-9	1.53-5	3.15-6	5.28-9	

DETAIL

Fig. 7.1c. Values of the modified error indicators $\eta^2(\Delta)$ for M = 26.

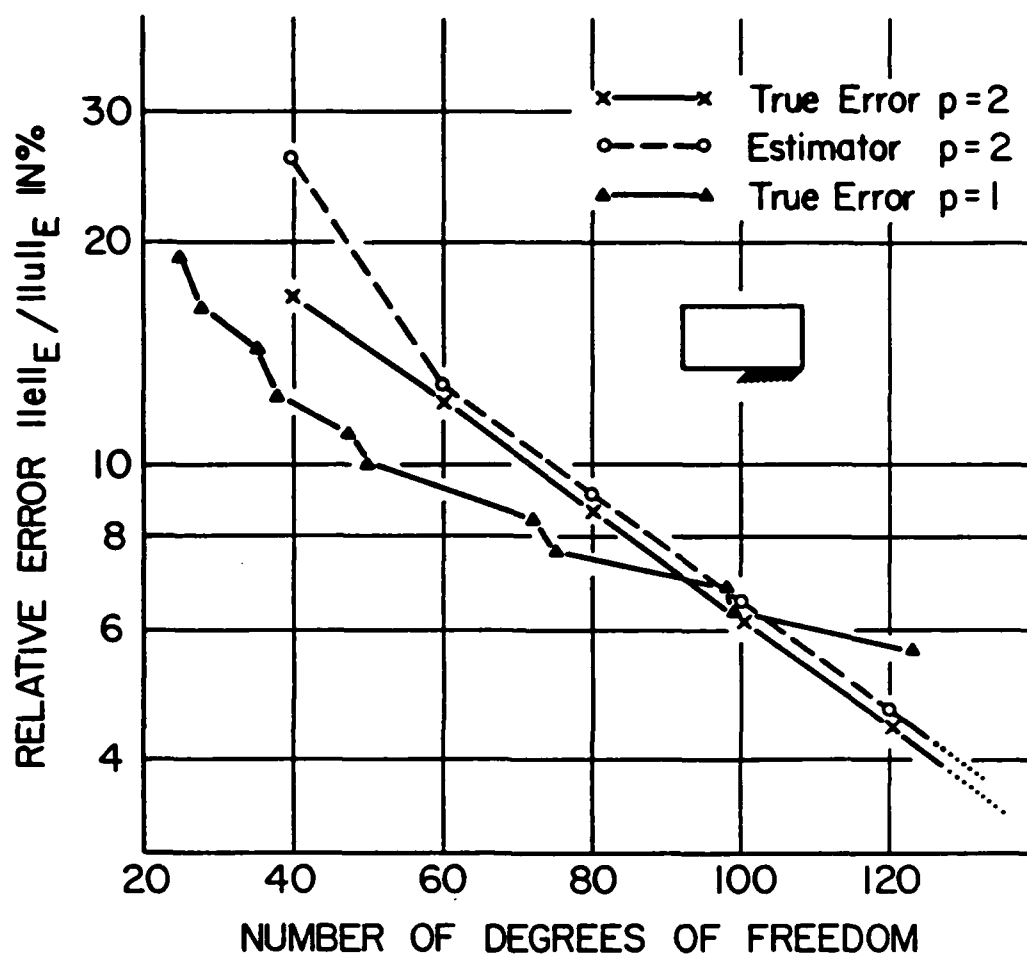


Fig. 7.2a. Accuracy of the finite element solution of Problem 1 for the element of degree $p = 1, 2$ (semilog scale).

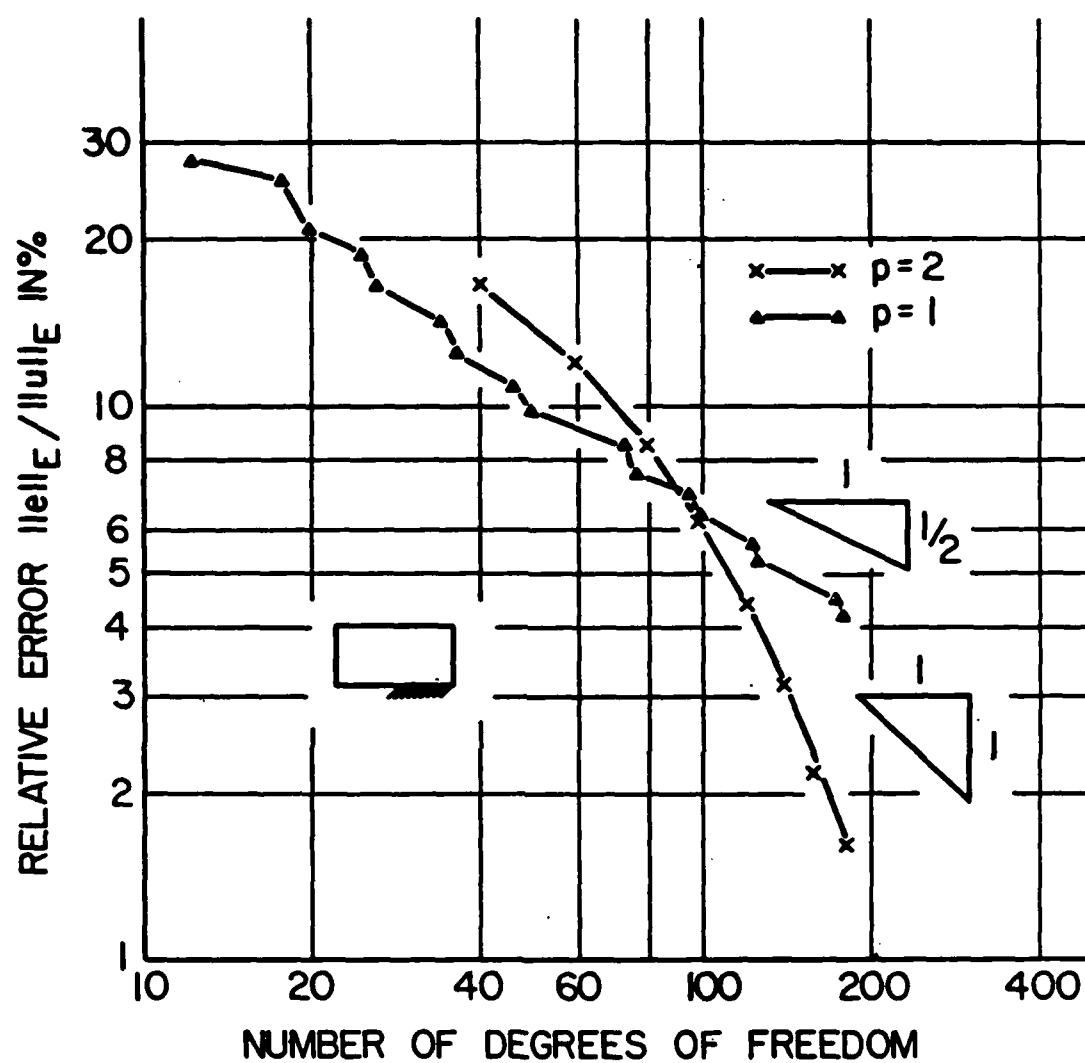


Fig. 7.2b. Accuracy of the finite element solution of Problem 1 for the elements of degree $p = 1, 2$ (log log scale).

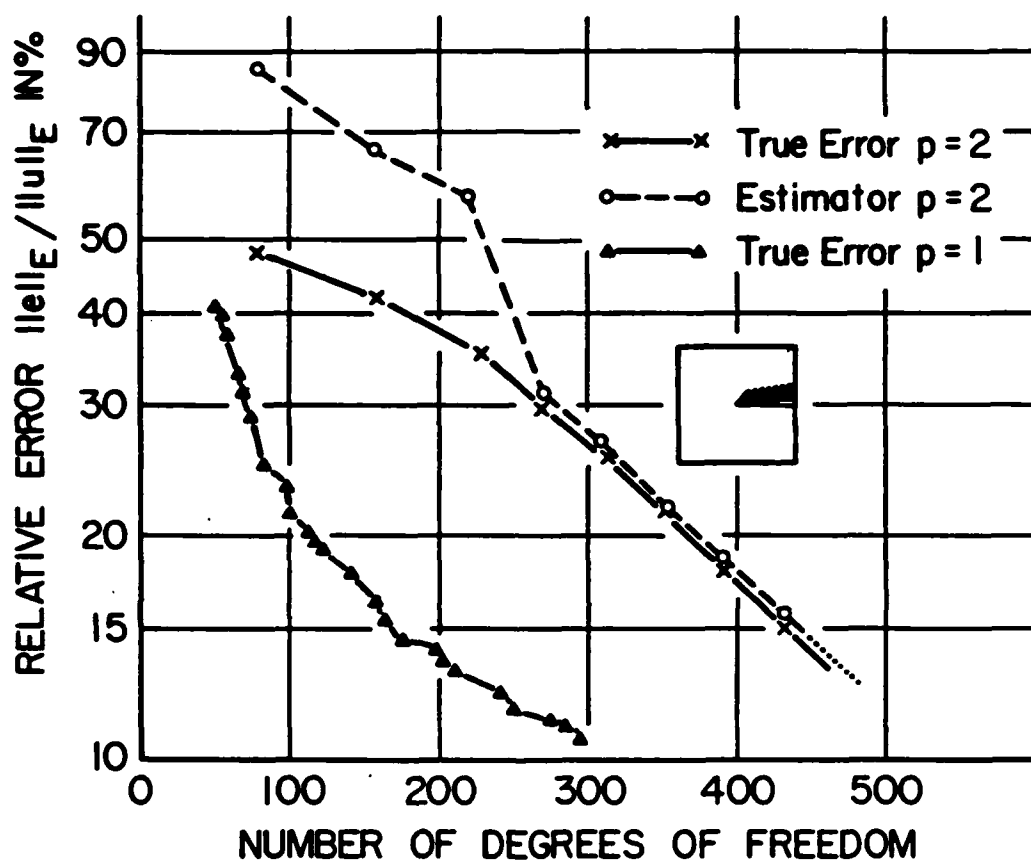


Fig. 7.3a. Accuracy of the finite element solution of Problem 2 for the elements of degree $p = 1, 2$ (semilog scale).

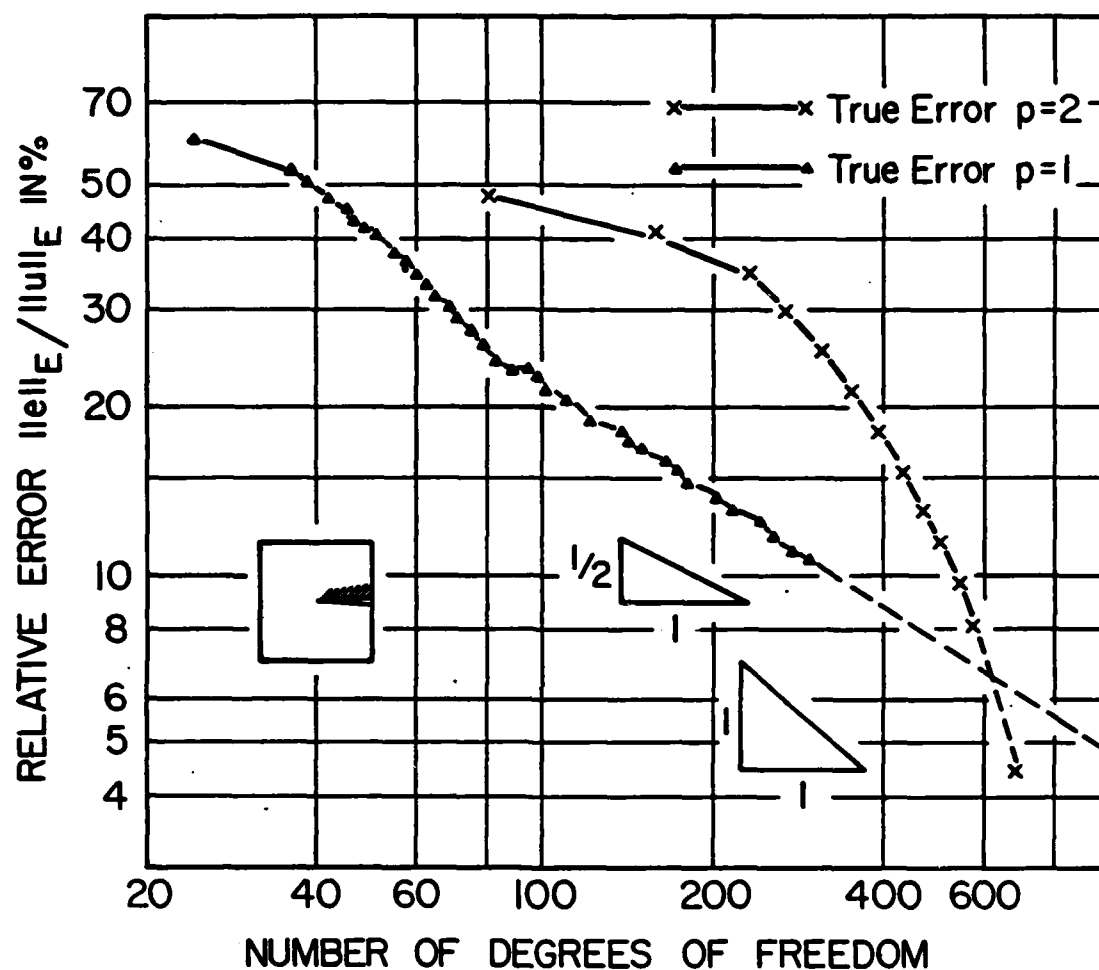


Fig. 7.3b. Accuracy of the finite element solution of Problem 2 for the elements of degree $p = 1, 2$ (log log scale).

TABLE 7.1

Number of elements M	Number of degrees of freedom N	Error indicator η^2 adjoined to singularity (modified)	Error $\ e\ _E$	Relative error $\ e\ _E/\ u\ _E \%$	Estimator ϵ (nonmodified)	Effectivity index θ of estimator ϵ	Estimator $\bar{\epsilon}$ (modified)	Effectivity index θ of estimator $\bar{\epsilon}$
8	40	.301-3 .267-4	.1116-1	16.97	.3865-2	.346	.1811-1	1.622
14	60	.591-4 .123-4	.7945-2	12.08	.2793-2	.351	.8475-2	1.067
20	80	.305-4 .627-5	.5566-2	8.602	.2041-2	.360	.6099-2	1.077
26	100	.153-4 .315-5	.4047-2	6.152	.1526-3	.377	.4350-2	1.074

8. CONCLUSIONS

We summarize now briefly our conclusions.

a) The error indicators have different structures for even and odd degree of elements. The even degrees are preferable from the implementational aspects.

b) The range of asymptotic exactness of the estimator has to be increased by the modification of the indicators in the terminal elements. The error estimates are then of high quality in the entire range of the accuracy.

c) Biquadratic elements lead often to an exponential convergence in the entire range of engineering accuracy.

d) The feedback algorithm should be based on a properly designed "short passes" which anticipate the pattern of the mesh refinement.

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Further information may be obtained from Professor I. Babuska, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

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